

# RATIONAL CURVES IN THE MODULI OF SUPERSINGULAR K3 SURFACES

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**Abstract.** We show how to construct non-isotrivial families of supersingular K3 surfaces over rational curves using a relative form the Artin-Tate isomorphism and twisted analogues of Bridgeland's results on moduli spaces of stable sheaves on elliptic surfaces. As a consequence, we show that every point of Artin invariant 10 in the Ogus space of marked supersingular K3 surfaces lies on infinitely many pairwise distinct rational curves canonically associated to elliptic structures on the underlying K3 surface.

## Contents

1. Introduction	1
1.1. Outline	2
1.2. History	2
1.3. Acknowledgments	2
2. A few remarks on the cohomology of $\mu_p$ on a family of curves	3
3. Singular fibers of elliptic K3 surfaces	4
4. Proof of Artin's result	5
5. Modular interpretation of the isomorphism $\mathrm{Br} \cong \mathrm{III}$	9
6. Rational curves in moduli spaces	19
7. Static pencils and deformations	21
References	26

## 1. Introduction

In this paper we study certain special rational curves in moduli spaces of K3 surfaces that are generated by cohomology classes. In particular, we focus on two results. Fix an algebraically closed field  $k$  of characteristic  $p > 0$  throughout this paper. Let (for now)  $\pi : X \rightarrow \mathrm{Spec} k$  be the structure map of a supersingular K3 surface.

**Theorem 1.1** (Artin). *The  $fppf$  sheaf  $\mathbf{R}^2\pi_*\mu_p$  is representable by a smooth group scheme over  $k$ , and the connected component  $\mathcal{C}^\circ$  of the identity is isomorphic to  $\mathbf{G}_a$ .*

This is a special case of a result of Artin, published as Theorem 3.1 of [2] with the caveat that the proof would be published elsewhere. I have not found the proof in the literature, so I provide a moduli-theoretic proof here. Using these “rational curves in cohomology,” we then prove the

following, which is essentially a relative version of the Artin-Tate isomorphism (Theorem 3.1 of [21], generalized in Proposition 4.5 of [5]).

**Theorem 1.2.** *Given a  $\mu_p$ -gerbe*

$$\mathcal{X} \rightarrow X \times \mathbf{A}^1$$

*inducing an isomorphism  $\mathbf{A}^1 \xrightarrow{\sim} \mathcal{C}^\circ$  (in Theorem 1.1) and a choice of elliptic fibration  $X \rightarrow \mathbf{P}^1$ , there is a canonically defined open substack of the stack of coherent  $\mathcal{X}$ -twisted sheaves that is a  $\mathbf{G}_m$ -gerbe over a non-isotrivial family  $Y \rightarrow \mathbf{A}^1$  of K3 surfaces. Moreover, the structure map of  $Y$  admits a factorization*

$$Y \rightarrow \mathbf{P}^1 \times \mathbf{A}^1$$

*such that for each geometric point  $t \rightarrow \mathbf{A}^1$ , the morphism*

$$Y_t \rightarrow \mathbf{P}_t^1$$

*is an étale form of*

$$X \otimes \kappa(t) \rightarrow \mathbf{P}^1 \otimes \kappa(t),$$

*and the fiber over  $0 \in \mathbf{A}^1$  is isomorphic to the original elliptic structure  $X \rightarrow \mathbf{P}^1$ . Finally, distinct elliptic structures on  $X$  give rise to distinct families of K3 surfaces  $Y$ .*

The family  $Y \rightarrow \mathbf{P}^1$  in Theorem 1.2 can be made to belong to various moduli problems (the Ogus space of marked K3 surfaces, the space of polarized K3 surfaces, etc.), at least over open subsets of the base  $\mathbf{A}^1$ . As a consequence, the formation of moduli spaces of twisted sheaves can be used to trace out rational curves on various moduli spaces using rational curves of cohomology classes.

**1.1. Outline.** In Sections 2 and 3 we prove a few preliminaries about the fppf cohomology of  $\mu_p$  on families of curves and the singular fibers of elliptic K3 surfaces. This is followed by a proof of Theorem 1.1 in Section 4 and a proof of Theorem 1.2 in Sections 5 and 6. Finally, in Section 7, we show that distinct elliptic pencils give rise to distinct rational curves in moduli.

We have included the bare minimum in this manuscript necessary to get the theory off the ground and provide an adequate reference for [11].

**1.2. History.** The work described here has been developing since 2011. I started giving public lectures about it in 2012 (see, for example, the Banff video

<http://videos.birs.ca/2012/12w5027/201203271601-Liebllich.mp4>

available since May of 2012) and discussed it by email and in person with various parties in 2012 and 2013. It was also described in my contribution to the 2012 Simons Symposium [10] (published in April of 2013).

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2. A few remarks on the cohomology of  $\mu_p$  on a family of curves

The following are very well known for étale cohomology with coefficients of order prime to the residue characteristic. They are also true for fppf cohomology with coefficients in  $\mu_p$ , as we record here.

**Lemma 2.1.** *Suppose  $C$  is a proper smooth curve over an algebraically closed field  $k$ . The Kummer sequence induces a canonical isomorphism*

$$\mathrm{Pic}(C)/p \mathrm{Pic}(C) = \mathbf{Z}/p\mathbf{Z} \xrightarrow{\sim} H^2(C, \mu_p)$$

*Proof.* By Tsen's theorem  $H^2(C, \mathbf{G}_m) = 0$ . Since the  $p$ th power Kummer sequence is exact on the fppf site, the result follows.  $\square$

**Proposition 2.2.** *Suppose  $Z \rightarrow G$  is a proper smooth morphism of finite presentation of relative dimension 1 with  $G$  connected and  $\alpha \in H^2(Z, \mu_p)$ . There exists a unique element  $a \in \mathbf{Z}/p\mathbf{Z}$  such that for every geometric point  $g \rightarrow G$ , the restriction  $\alpha_{Z_g} \in \mathbf{Z}/p\mathbf{Z}$  is equal to  $a$  via the isomorphism of Lemma 2.1.*

*Proof.* It suffices to prove this under the assumption that  $G$  is the spectrum of a complete dvr.

**Lemma 2.3.** *When  $G$  is the spectrum of a complete dvr, we have that  $H^2(Z, \mathbf{G}_m) = 0$ .*

*Proof.* First, since  $Z$  is regular the group is torsion. Thus, any  $\mathbf{G}_m$ -gerbe is induced by a  $\mu_n$ -gerbe for some  $n$ . Fix a  $\mu_n$ -gerbe  $\mathcal{Z} \rightarrow Z$ . By Tsen's theorem and Lemma 3.1.1.8 of [8], there is an invertible  $\mathcal{Z}_g$ -twisted sheaf  $L$ , where  $g$  is the closed point of  $G$ . The obstruction to deforming such a sheaf lies in

$$H^2(Z, \mathcal{O}) = 0,$$

so that  $L$  has a formal deformation over the completion of  $\mathcal{Z}$ . By the Grothendieck Existence Theorem for proper Artin stacks, Theorem 11.1 of [18] (or, in this case, the classical Grothendieck Existence Theorem for coherent modules over an Azumaya algebra representing  $\alpha$ ), this formal deformation algebraizes, trivializing the class of  $\mathcal{Z}$  in  $H^2(Z, \mathbf{G}_m)$ , as desired.  $\square$

Applying the lemma and the Kummer sequence, we see that

$$\mathrm{Pic}(Z)/p \mathrm{Pic}(Z) \xrightarrow{\sim} H^2(Z, \mu_p).$$

But restricting to a fiber defines a canonical isomorphism

$$\mathrm{Pic}(Z)/p \mathrm{Pic}(Z) \xrightarrow{\sim} \mathrm{Pic}(Z_g)/p \mathrm{Pic}(Z_g) \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}$$

independent of the point  $g$ . The result follows.  $\square$

**Corollary 2.4.** *Suppose  $E \subset Z \rightarrow T$  is a family of smooth genus 1 fibers in a proper flat family of elliptic surfaces of finite presentation over a connected base. Given a class*

$$\alpha \in H^2(Z, \mu_p),$$

*there is an element  $a \in \mathbf{Z}/p\mathbf{Z}$  such that for every geometric point  $t \in T$ , the restriction of  $\alpha$  to  $E_t$  equals  $a$  via the isomorphism of Lemma 2.1.*

**Example 2.5.** This kind of thing is not as utterly trivial as it seems. Consider the cuspidal cubic  $C$ . The Kummer sequence shows that there is a class  $\alpha \in H^2(C \times \mathbf{A}^1, \mu_p)$  whose value over 0 is trivial and whose value over any other geometric point is non-trivial. In fact, there is an isomorphism of fppf functors

$$R^2 f_* \mu_p \xrightarrow{\sim} G_a$$

where  $f : C \rightarrow \operatorname{Spec} k$  denotes the structure map. When multiplication by  $p$  on the Picard scheme is ramified (or trivial, as in the case of  $G_a$ ), interesting behavior is possible. This makes the study of stable twisted sheaves of rank 1 on families of  $\mu_p$ -gerbes on elliptic surfaces somewhat interesting.

### 3. Singular fibers of elliptic K3 surfaces

I am indebted to Aise Johan de Jong for pointing out an error in an earlier version of this section, for telling me about extremal elliptic surfaces, and for suggesting the idea of using  $k$ -rational  $j$ -invariants to augment the locus of singular fibers in the extremal case (as used in Corollary 5.2 below).

Let  $X \rightarrow \mathbf{P}^1$  be a proper morphism from a K3 surface with smooth geometrically connected geometric fiber of dimension 1. Recall that there is an associated Jacobian fibration  $\pi : J \rightarrow \mathbf{P}^1$  (see, e.g., Section 4 of Chapter 11 of [6]) that is also a K3 surface, but that possesses a section  $\sigma : \mathbf{P}^1 \rightarrow J$ . Using the classification of singular fibers and relative minimality of elliptic fibrations on K3 surfaces, we know that there is an étale surjection  $U \rightarrow \mathbf{P}^1$  such that  $X_U$  and  $J_U$  are isomorphic (see Corollary 5.5 of Chapter 11 of [6]). In particular,  $X$  and  $J$  are fiberwise isomorphic.

We will fix  $X \rightarrow \mathbf{P}^1$  and  $J \rightarrow \mathbf{P}^1$  in what follows. Recall the Shioda-Tate formula: if  $r_t$  is the number of irreducible components of the fiber of  $\pi$  over  $t$ , then

$$\rho(J) = 2 + \sum_t (r_t - 1) + \operatorname{rk} J(k(t)),$$

the last term being the rank of the Mordell-Weil group, the group of rational points on the generic fiber of  $\pi$ .

**Proposition 3.1.** *Either  $\pi : J \rightarrow \mathbf{P}^1$  has at least 3 singular fibers or the  $j$ -invariant of  $J_\eta$  does not lie in  $k \subset k(t)$ .*

*Proof.* Suppose there are  $m$  singular fibers  $F_1, \dots, F_m$ . The Kodaira classification of singular fibers of minimal elliptic fibrations (Theorem IV.8.2 of [20]) shows that every singular fiber  $F_i$  with  $n_i$  components has  $\ell$ -adic Euler characteristic

$$\chi(F, \mathbf{Z}_\ell) \leq n_i + 1.$$

Since a smooth curve of genus 1 has Euler characteristic 0, we have two inequalities

$$2 - 2m + \sum_{i=1}^m (n_i + 1) = \rho(X) - \operatorname{rk} J(k(t)) \leq 22$$

$$\sum_{i=1}^m \chi(F_i, \mathbf{Z}_\ell) = 24 \leq \sum_{i=1}^m (n_i + 1),$$

the first coming from the Shioda-Tate formula. Letting  $S = \sum (n_i + 1)$ , this yields

$$24 \leq S \leq 22 + 2m.$$

We conclude that  $m \geq 2$ , and if  $m = 2$  then we must have  $\operatorname{rk} J(k(t)) = 0$ . In other words, when  $m = 2$  the surface  $J$  is extremal. By Theorem 6.1(1) of [19] (which, in spite of the paper's title, does not assume that the base field has characteristic 2 or 3), the latter implies that the  $j$ -invariant of  $J_{k(t)}$  does not lie in  $k$ , for otherwise  $J$  would be rational (and we know it is a K3 surface).  $\square$

**Corollary 3.2.** *Let  $X$  be a supersingular K3 surface and  $f : X \rightarrow \mathbf{P}^1$  a morphism with smooth connected geometric generic fiber of genus 1. Then either*

- (1)  *$f$  has at least three singular fibers, or*
- (2) *for any algebraically closed extension field  $k \subset K$  and any element  $\alpha \in k$ , the set of points  $t \in \mathbf{P}^1(K)$  such that  $j(X_t) = \alpha$  lies in the image of the extension of scalars map  $\mathbf{P}^1(k) \hookrightarrow \mathbf{P}^1(K)$ .*

*Proof.* As noted in the first paragraph of this section, the singular fibers of  $f$  are the same as the singular fibers of the Jacobian fibration  $\pi : J \rightarrow \mathbf{P}^1$ , and for any geometric point  $t \rightarrow \mathbf{P}^1$  that avoids singular fibers, we have  $j(X_t) = j(J_t)$ . By Proposition 3.1, if  $\pi$  has only two singular fibers then the  $j$ -invariant map

$$j : \mathbf{P}^1 \dashrightarrow \mathbf{A}^1$$

is non-constant, hence quasi-finite. But if  $Z \rightarrow W$  is any quasi-finite morphism of schemes of finite type over an algebraically closed field  $k$  and  $k \subset K$  is an algebraically closed extension field, the diagram

$$\begin{array}{ccc} Z(k) & \longrightarrow & W(k) \\ \downarrow & & \downarrow \\ Z(K) & \longrightarrow & W(K) \end{array}$$

is Cartesian. Indeed, the fiber of  $Z \rightarrow W$  over a  $k$ -point  $p \rightarrow W$  is a finite  $k$ -scheme, whose reduced structure must be isomorphic to a disjoint union of copies of  $\operatorname{Spec} k$  by the structure theory of finite-dimensional algebras over a field.  $\square$

#### 4. Proof of Artin's result

Fix a supersingular K3 surface  $X$  over  $k$  and let  $\pi : X \rightarrow \operatorname{Spec} k$  denote the structure morphism. In this section, we prove Theorem 1.1 using the stack of Azumaya algebras on  $X$ .

**Theorem 4.1.** *The big fppf sheaf  $\mathbf{R}^2\pi_*\mu_p$  is representable by a smooth group scheme over  $k$  whose connected component is isomorphic to  $\mathbf{G}_a$ .*

We will write  $\mathcal{C} = \mathbf{R}^2\pi_*\mu_p$  in the remainder of this section.

**Proposition 4.2.** *The diagonal*

$$\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$$

*is representable by closed immersions of finite presentation.*

*Proof.* Let  $a, b : T \rightarrow \mathcal{C}$  be two maps corresponding to classes

$$a, b \in H^0(T, \mathbf{R}^2\pi_*\mu_p).$$

To prove that the locus where  $a = b$  is represented by a closed subscheme of  $T$ , it suffices by translation to assume that  $b = 0$  and prove that the functor  $Z(a)$  sending a  $T$ -scheme  $S \rightarrow T$  to  $\emptyset$  if  $a_S \neq 0$  and  $\{\emptyset\}$  otherwise is representable by a closed subscheme  $Z_a \subset T$ .

First, suppose  $a$  is the image of a class  $\alpha \in H^2(X_T, \mu_p)$  corresponding to a  $\mu_p$ -gerbe  $\mathcal{X} \rightarrow X_T$ . Since  $\mathbf{R}^1\pi_*\mu_p = 0$ , we see that the functor  $Z(a)$  parametrizes schemes  $S \rightarrow T$  such that there is an  $\alpha' \in H^2(S, \mu_p)$  with

$$\alpha'|_{X_S} = \alpha|_{X_S}.$$

Let  $\mathcal{P} \rightarrow T$  be the stack whose objects over  $S \rightarrow T$  are families of  $\mathcal{X}_S$ -twisted invertible sheaves  $\mathcal{L}$  together with isomorphisms  $\mathcal{L}^{\otimes p} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_S}$ . The stack  $\mathcal{P}$  is a  $\mu_p$ -gerbe over a quasi-separated algebraic space  $P \rightarrow T$  that is locally of finite presentation (Proposition 2.3.1.1 of [7] and Section C.23 of [1]). Moreover, since  $\text{Pic}_X$  is torsion free, the natural map  $P \rightarrow T$  is a monomorphism. Note that if we change  $\alpha$  by the preimage of a class  $\alpha' \in H^2(T, \mu_p)$  we do not change  $P$  (but we do change the class of the gerbe  $\mathcal{P} \rightarrow P$  by  $\alpha'$ ).

**Lemma 4.3.** *The algebraic space  $P \rightarrow T$  is a closed immersion of finite presentation.*

*Proof.* First, let us show that  $P$  is of finite presentation. It suffices to show that  $P$  is quasi-compact under the assumption that  $T$  is affine. Moreover, since  $\mathcal{C}$  is locally of finite presentation, we may assume that  $T$  is Noetherian. By Gabber's Theorem (Theorem 1.1 of [4]) there is a Brauer-Severi scheme  $V \rightarrow X$  such that  $\alpha|_V$  has trivial Brauer class, i.e., so that there is an invertible sheaf  $L \in \text{Pic}(V)$  satisfying

$$\alpha|_V = [L]^{1/p} \in H^2(V, \mu_p),$$

where  $[L]^{1/p}$  denotes the  $\mu_p$ -gerbe over  $V$  parametrizing  $p$ th roots of  $L$ . Writing

$$\mathcal{V} = \mathcal{X} \times_{X_T} V,$$

we know from the isomorphism  $\mathcal{V} \cong [L]^{1/p}$  that there is an invertible  $\mathcal{V}$ -twisted sheaf  $\mathcal{L}$  such that

$$\mathcal{L}^{\otimes p} \cong L.$$

Let  $W$  denote the algebraic space parametrizing invertible  $\mathcal{V}$ -twisted sheaves whose  $p$ th tensor powers are trivial. By the argument in the preceding paragraph, tensoring with  $\mathcal{L}^\vee$  defines an isomorphism between  $W$  and the fiber of the  $p$ th power map

$$\mathrm{Pic}_{V/T} \rightarrow \mathrm{Pic}_{V/T}$$

over  $[L^\vee]$ . Since the  $p$ th power map is a closed immersion ( $X$  being K3), we see that  $W$  is of finite type. The following lemma then applies to show that  $P$  is of finite type.

**Lemma 4.4.** *The pullback map  $\mathrm{Pic}_{\mathcal{X}/T}^{\mathrm{tw}} \rightarrow \mathrm{Pic}_{\mathcal{V}/T}^{\mathrm{tw}}$  is of finite type.*

*Proof.* It suffices to prove the corresponding results for the stacks of invertible twisted sheaves. Since  $V \times_X V$  and  $V \times_X V \times_X V$  are proper over  $T$  this follows from descent theory: the category of invertible  $\mathcal{X}$ -twisted sheaves is equivalent to the category of invertible  $\mathcal{V}$ -twisted sheaves with a descent datum on  $\mathcal{V} \times_{\mathcal{X}} \mathcal{V}$ . Thus, the fiber over  $L$  on  $\mathcal{V}$  is a locally closed subspace of

$$\mathrm{Hom}_{\mathcal{V} \times_{\mathcal{X}} \mathcal{V}}(\mathrm{pr}_1^* L, \mathrm{pr}_2^* L).$$

Since the latter is of finite type (in fact, a cone in an affine bundle), the result follows.  $\square$

We claim that  $P$  is proper over  $T$ . To see this, we may use the fact that it is of finite presentation (and everything is of formation compatible with base change) to reduce to the case in which  $T$  is Noetherian, and then we need only check the valuative criterion over dvrs. Thus, suppose  $E$  is a dvr with fraction field  $F$  and  $p : \mathrm{Spec} F \rightarrow P$  is a point. Replacing  $E$  by a finite extension, we may assume that  $p$  comes from an invertible  $\mathcal{X}_F$ -twisted sheaf  $\mathcal{L}$ . Taking a reflexive extension and using the fact that  $\mathcal{X}_E$  is locally factorial, we see that  $\mathcal{L}$  extends to an invertible  $\mathcal{X}_E$ -twisted sheaf  $\mathcal{L}_E$ . Since  $\mathrm{Pic}_X$  is separated, it follows that  $\mathcal{L}_E$  induces the unique point of  $P$  over  $E$  inducing  $p$ .

Since a proper monomorphism is a closed immersion, we are done.  $\square$

We now claim that the locus  $Z(a) \subset T$  is represented by the closed immersion  $P \rightarrow T$ .

**Lemma 4.5.** *A class  $\alpha \in H^2(X_T, \mu_p)$  represented by a  $\mu_p$ -gerbe  $\mathcal{X} \rightarrow X_T$  is trivial if and only there is an invertible  $\mathcal{X}$ -twisted sheaf  $\mathcal{L}$  such that*

$$\mathcal{L}^{\otimes p} \cong \mathcal{O}_{\mathcal{X}}.$$

*Proof.* Given a scheme  $S$ , the gerbe  $\mathrm{B}\mu_{p,S} \rightarrow S$  represents the stack of pairs  $(\mathcal{L}, \varphi)$  with  $\mathcal{L}$  an invertible sheaf and  $\varphi : \mathcal{L}^{\otimes p} \xrightarrow{\sim} \mathcal{O}$  is a trivialization. By definition,  $\alpha$  is trivial if and only if there is an isomorphism of stacks  $\mathcal{X} \xrightarrow{\sim} \mathrm{B}\mu_{p,X_T}$ . The result follows.  $\square$

Now consider the  $\mu_p$ -gerbe  $\mathcal{P} \rightarrow P$ . Subtracting the pullback from  $\alpha$  yields a class such that the associated Picard stack  $\mathcal{P}' \rightarrow P$  is trivial, whence there is an invertible twisted sheaf with trivial  $p$ th power. In other words,

$$a|_P = 0 \in \mathcal{C}(P).$$

On the other hand, if  $a|_S = 0$  then up to changing  $a$  by the pullback of a class from  $S$ , there is an invertible twisted sheaf with trivial  $p$ th power. But this says precisely that  $S$  factors through the moduli space  $P$ , as desired.  $\square$

Let  $\underline{Az}$  be the  $k$ -stack whose objects over  $T$  are Azumaya algebras  $\mathcal{A}$  of degree  $p$  on  $X_T$  such that for every geometric point  $t \rightarrow T$ , we have

$$\ker(\mathrm{Tr} : H^2(X, \mathcal{A}) \rightarrow H^2(X, \mathcal{O})) = 0.$$

It is well known that  $\underline{Az}$  is an Artin stack locally of finite type over  $k$  (see, for example, Lemma 3.3.1 of [9], and note that the trace condition is open). There is a morphism of stacks

$$\chi : \underline{Az} \rightarrow \mathbf{R}^2\pi_*\mu_p$$

(with the latter viewed as a stack with no non-trivial automorphisms in fiber categories) given as follows. Any family  $\mathcal{A} \in \underline{Az}_T$  has a corresponding class

$$[\mathcal{A}] \in H^1(X_T, \mathrm{PGL}_p).$$

The non-Abelian coboundary map yields a class in  $H^2(X_T, \mu_p)$  which has a canonical image

$$\chi(\mathcal{A}) \in H^0(T, \mathbf{R}^2\pi_*\mu_p).$$

**Proposition 4.6.** *The morphism  $\chi$  described above is representable by smooth Artin stacks.*

*Proof.* Since  $\underline{Az}$  is locally of finite type over  $k$  and the diagonal of  $\mathcal{C}$  is representable by closed immersions of finite presentation, it suffices to show that  $\chi$  is formally smooth. Suppose  $A' \rightarrow A$  is a square-zero extension of Noetherian rings and consider a diagram of solid arrows

$$\begin{array}{ccc} \mathrm{Spec} A' & \longrightarrow & \underline{Az} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \mathrm{Spec} A & \longrightarrow & \mathcal{C}. \end{array}$$

We wish to show that we can produce the dashed diagonal arrow. Define a stack  $\mathcal{S}$  on  $\mathrm{Spec} A$  whose objects over an  $A$ -scheme  $U \rightarrow \mathrm{Spec} A$  are dashed arrows in the restricted diagram

$$\begin{array}{ccc} U' & \longrightarrow & \underline{Az} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ U & \longrightarrow & \mathcal{C}, \end{array}$$

where  $U' = U \otimes_A A'$ , and whose morphisms are isomorphisms between the objects of  $\underline{Az}$  over  $U$  restricting to the identity on the restrictions to  $U'$ .

**Claim 4.7.** *The stack  $\mathcal{S} \rightarrow \mathrm{Spec} A$  is an fpf gerbe with coherent band.*

*Proof.* First, it is clear that  $\mathcal{S}$  is locally of finite presentation. Suppose  $U$  is the spectrum of a complete local Noetherian ring with algebraically closed residue field. Then  $H^2(U, \mu_p) = 0$  and the section  $U \rightarrow \mathcal{C}$  is equivalent to a class

$$\alpha \in H^2(X_U, \mu_p).$$



Let  $\mathcal{X} \rightarrow X_U$  be a  $\mu_p$ -gerbe representing  $\alpha$  and write  $\mathcal{X}' = \mathcal{X} \otimes_A A'$  the restriction of  $\mathcal{X}$  to  $U'$ . An object of  $\underline{A}_{\mathbb{Z}_{U'}}$  is then identified with  $\mathcal{E}nd(V)$  where  $V$  is a locally free  $\mathcal{X}'$ -twisted sheaf of rank  $p$  with trivial determinant. The obstruction to deforming such a sheaf lies in

$$\ker(H^2(X_{U'}, \mathcal{E}nd(V) \otimes I) \rightarrow H^2(X_{U'}, \mathcal{O} \otimes I)),$$

and deformations are a pseudo-torsor under

$$\ker(H^1(X_{U'}, \mathcal{E}nd(V) \otimes I) \rightarrow H^1(X_{U'}, \mathcal{O} \otimes I)) = H^1(X_{U'}, \mathcal{E}nd(V) \otimes I).$$

Standard arguments starting from the assumption on the geometric points of  $\underline{A}_{\mathbb{Z}}$  show that the obstruction group is trivial, while the band is the coherent sheaf  $\mathbf{R}^1\pi_*\mathcal{E}nd(V)$ , as desired.  $\square$

Since any gerbe with coherent band over an affine scheme is neutral, we conclude that  $\mathcal{S}$  has a section. In other words, a dashed arrow exists, as desired.  $\square$

*Proof of Theorem 4.1.* Using Proposition 4.6, a smooth cover  $B \rightarrow \underline{A}_{\mathbb{Z}}$  gives rise to a smooth cover  $B \rightarrow \mathcal{C}$ . Thus, by Proposition 4.2,  $\mathcal{C}$  is a separated algebraic group-space locally of finite type. Since  $k$  is algebraically closed, it follows that  $\mathcal{C}$  is in fact a group scheme locally of finite type. Finally, since  $X$  is supersingular we have that the completion of  $\mathcal{C}$  at the identity section is isomorphic to  $\widehat{\mathbf{G}}_a$ , which is formally smooth and  $p$ -torsion. It follows that  $\mathcal{C}$  is smooth over  $k$  with 1-dimensional  $p$ -torsion connected component. The only  $p$ -torsion smooth 1-dimensional  $k$ -group scheme is  $\mathbf{G}_a$ , completing the proof.  $\square$

### 5. Modular interpretation of the isomorphism $\mathrm{Br} \cong \mathrm{III}$

Fix an elliptic K3 surface  $f : X \rightarrow \mathbf{P}^1$  over the algebraically closed field  $k$ . Given a  $\mu_n$ -gerbe  $\mathcal{X} \rightarrow X$ , let

$$a(\mathcal{X}) \in \frac{1}{n}\mathbf{Z}/\mathbf{Z}$$

be the unique element that corresponds to the cohomology class of the restriction of  $\mathcal{X}$  to any smooth fiber  $E \subset X$  of  $f$  under the natural isomorphism

$$\frac{1}{n}\mathbf{Z}/\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}.$$

Fix an ample divisor  $H \subset X$ .

Recall the following special case of a theorem of Artin and Tate (Theorem 3.1 of [21]).

**Theorem 5.1** (Artin-Tate). *The edge map in the  $E^2$  term of the Leray spectral sequence for  $\mathbf{G}_m$  on  $f : X \rightarrow \mathbf{P}^1$  yields an isomorphism*

$$\mathrm{Br}(X) \rightarrow \mathrm{III}(k(t), \mathrm{Pic}_{X_{k(t)}/k(t)}),$$

*resulting in a natural surjection*

$$\mathrm{III}(k(t), \mathrm{Jac}(X_{k(t)})) \twoheadrightarrow \mathrm{Br}(X)$$

*with kernel isomorphic to  $\mathbf{Z}/i\mathbf{Z}$ , where  $i$  is the index of the generic fiber  $X_{k(t)}$  over  $k(t)$ .*

In particular, if  $X \rightarrow \mathbf{P}^1$  has a section, the latter arrow is an isomorphism.

*Proof.* This is Proposition 4.5 (and “cas particulier (4.6)” of [5]).  $\square$

It follows by descent theory that any element of  $\text{III}(k(t), \text{Jac}(X_{k(t)}))$  corresponds to an étale form  $X'$  of  $X$ , and  $X'$  is also a K3 surface. In this section we will describe this isomorphism geometrically using the theory of stable twisted sheaves. This geometric description will allow us to take a varying Brauer class on  $X$  and produce a family of K3 surfaces (that are each forms of a given elliptic fibration on  $X$ ) in Section 6. The central interest arises from the following corollary.

**Corollary 5.2.** *Let  $K/k$  be an algebraically closed extension field, and suppose  $\alpha \in \text{Br}(X_K)$  is not in the image of the restriction map*

$$\text{Br}(X) \rightarrow \text{Br}(X_K).$$

*No étale form  $X'$  of  $X_K$  mapping to  $\alpha$  via Theorem 5.1 is defined over  $k$ .*

*Proof.* By functoriality of the Leray spectral sequence, the diagram

$$(5.0.1) \quad \begin{array}{ccc} \text{III}(K(t), \text{Jac}(X_{K(t)})) & \longrightarrow & \text{Br}(X_K) \\ \uparrow & & \uparrow \\ \text{III}(k(t), \text{Jac}(X_{k(t)})) & \longrightarrow & \text{Br}(X) \end{array}$$

commutes.

Since  $X' \rightarrow \mathbf{P}_K^1$  is a form of  $X_K \rightarrow \mathbf{P}_K^1$ , we know by Corollary 3.2 that either

- (1)  $X' \rightarrow \mathbf{P}_K^1$  has at least three singular fibers located over the image of the map

$$\mathbf{P}^1(k) \rightarrow \mathbf{P}^1(K),$$

or

- (2) for any  $\alpha \in k$ , the set of points  $t \in \mathbf{P}^1(K)$  such that  $j(X'_t) = \alpha$  lies in the image of

$$\mathbf{P}^1(k) \hookrightarrow \mathbf{P}^1(K).$$

If  $X'$  is defined over  $k$  then so is any divisor class, and thus the elliptic fibration

$$X' \rightarrow \mathbf{P}_K^1$$

is a  $K$ -linear change of basis from an elliptic fibration

$$X'_0 \rightarrow \mathbf{P}^1$$

over  $k$ . On the other hand, by condition (1) or condition (2) above, the change of basis must send at least three elements of  $\mathbf{P}^1(k) \subset \mathbf{P}^1(K)$  into  $\mathbf{P}^1(k)$ . We conclude that the change of basis transforming  $X'_0$  into  $X'$  is  $k$ -linear (as any change of basis is determined by its action on three points, and any injective map on three  $k$ -points determines a  $k$ -linear change of basis). In other words,  $X' \rightarrow \mathbf{P}_K^1$  is isomorphic as an elliptic fibration to the base change of an elliptic fibration over  $k$ , which we may assume without loss of generality is  $X'_0 \rightarrow \mathbf{P}^1$ .

Moreover, if  $X' \rightarrow \mathbf{P}_K^1$  is a form of  $X_K \rightarrow \mathbf{P}_K^1$  then  $X'_0 \rightarrow \mathbf{P}^1$  must be a form of  $X \rightarrow \mathbf{P}^1$ , since the epimorphism property of the big étale  $k$ -sheaves

$$\mathrm{Isom}_{\mathbf{P}^1}(X, X'_0) \rightarrow \mathbf{P}_k^1$$

can be detected after the base change via  $k \hookrightarrow K$ . It then follows from diagram (5.0.1) that  $\alpha$  must be in the image of the restriction map, which is a contradiction.  $\square$

*Remark 5.3.* The moral of the preceding results: a moving Brauer class gives rise to a moving family of K3 surfaces (and not merely a moving family of elliptic pencils), at least rationally, i.e., when the base of the family is the spectrum of a field. (Note that when  $X$  does not have a section, there is some ambiguity about this family of torsors; if we only work with torsors that are deformations of the trivial torsor this goes away.) We will now show that this is true in a strong sense by showing that the Artin-Tate isomorphism can be made regular over a  $k$ -scheme, using the moduli of twisted sheaves.

Let  $\mathrm{CH}(X)$  denote the (graded) Chow group of algebraic cycles on  $X$  up to numerical equivalence. The fundamental class maps define embeddings of  $\mathrm{CH}(X)$  into  $H(X)$ , where  $H(X)$  is any of the “usual” integral cohomology theories (crystalline,  $\ell$ -adic).

**Definition 5.4.** Given a perfect complex  $F$  of  $\mathcal{X}$ -twisted sheaves, the *twisted Chern character* of  $F$  is

$$\mathrm{ch}(F) := \sqrt[n]{\mathbf{R}\gamma_*(F^{\mathbf{L}}_{\otimes n})} \in \mathrm{CH}(X) \otimes \mathbf{Q}.$$

The *twisted Mukai vector* of  $F$  is

$$v(F) := \mathrm{ch}(F) \sqrt{\mathrm{Td}_X}.$$

It is well known that given a pair of perfect complexes  $F$  and  $G$  of  $\mathcal{X}$ -twisted sheaves, the Riemann-Roch theorem holds:

$$\chi(F, G) = \deg(\mathrm{ch}(F^\vee \otimes^{\mathbf{L}} G) \cdot \mathrm{Td}_X).$$

Moreover, the twisted Mukai vector is locally constant in a family of perfect complexes on  $\mathcal{X}$  (cf. the following discussion and Proposition 2.2.7.22 of [7]).

Recall the following definition, Definition 2.2.7.6 of [7].

**Definition 5.5.** Given a perfect complex  $F$  of  $\mathcal{X}$ -twisted sheaves, the *geometric Hilbert polynomial* of  $F$  is the function  $P_F(m) = \deg(\mathrm{ch}(F(m)) \cdot \mathrm{Td}_X)$ .

As explained in Section 2.2.7.5 of [7],  $P_F$  is a numerical polynomial with the usual properties. In particular, we can use it to define stability and semistable of sheaves. Write  $p_F$  for the *reduced Hilbert polynomial* given by

$$p_F(m) = \frac{1}{\alpha_d} P_F(m),$$

where  $\alpha_d$  is the leading coefficient of  $P_F$ . (See Definition 2.3.2.3 of [7].)

**Definition 5.6.** A pure  $\mathcal{X}$ -twisted sheaf  $F$  is *stable* if for every subsheaf  $G \subset F$  we have that

$$p_G(m) < p_F(m)$$

for all  $m$  sufficiently large.

**Example 5.7.** If  $F$  is an invertible  $\mathcal{X}$  sheaf supported on a smooth curve in  $X$  then  $F$  is stable with respect to any polarization.

Just as in the classical case of (untwisted) elliptic surfaces, we will produce a form of  $X$  by taking a moduli space. Recall in what follows that  $\mathcal{X}$  is a  $\mu_n$ -gerbe (to remind us of what  $n$  means!).

**Definition 5.8.** Let  $\mathcal{M}_{\mathcal{X}}$  be the  $k$ -stack whose objects over  $T$  are  $T$ -flat quasi-coherent  $\mathcal{X}_T$ -twisted sheaves  $F$  of finite presentation such that for each geometric point  $t \in T$ , the fiber  $F_t$  has twisted Mukai vector  $(0, \mathcal{O}(E), na(\mathcal{X}) - 1)$  and is  $H$ -slope-stable.

In particular, each fiber sheaf  $F_t$  above is required to be pure (part of slope-stability), necessarily of dimension 1.

**Definition 5.9.** We define two relative stacks.

- (1) Let  $\mathcal{R}_{\mathcal{X}}^{\text{big}} \rightarrow \mathbf{P}^1$  be the stack whose objects over  $T \rightarrow \mathbf{P}^1$  are  $T$ -flat quasi-coherent  $\mathcal{X} \times_{\mathbf{P}^1} T$ -twisted sheaves  $F$  of finite presentation such that for each geometric point  $t \in T$ , the pushforward of the fiber  $F_t$  along the natural closed immersion

$$\mathcal{X} \times_{\mathbf{P}^1} t \rightarrow \mathcal{X} \times t$$

is  $H$ -slope stable with twisted Chern class  $(0, \mathcal{O}(E), na(\mathcal{X}) - 1)$ .

- (2) Let  $\mathcal{R}_{\mathcal{X}}$  for the reduced closed substack given by the closure of the preimage of the generic point of  $\mathbf{P}^1$ .

By the usual results on stability (summarized in Section 3.2.1 of [8]),  $\mathcal{M}_{\mathcal{X}}$  is a  $\mathbf{G}_m$ -gerbe over an algebraic space  $M_{\mathcal{X}}$  and  $\mathcal{R}_{\mathcal{X}}$  is a  $\mathbf{G}_m$ -gerbe over an algebraic space  $R_{\mathcal{X}} \rightarrow \mathbf{P}^1$ .

*Remark 5.10.* The reader will note that we define the stability condition in terms of the pushforward of the family to  $\mathcal{X}$ , rather than in the usual classical way, in terms of a relative polarization on  $X$  over  $\mathbf{P}^1$ . This is done in order to avoid dealing with Hilbert polynomials on gerbes – which are not purely cohomological in nature – in the case of a singular variety (such as a singular fiber of the pencil).

**Example 5.11.** When  $\mathcal{X} \rightarrow X$  is the trivial gerbe  $X \times \mathbf{B}\mu_n$ , we can compare this to a classical moduli problem. There is an invertible  $\mathcal{X}$ -twisted sheaf (corresponding to the natural inclusion character  $\mu_n \hookrightarrow \mathbf{G}_m$ )  $\mathcal{L}$  such that  $\mathcal{L}^{\otimes n} \cong \mathcal{O}_{\mathcal{X}}$ . Tensoring with  $\mathcal{L}^{\vee}$  and pushing forward to  $X$  defines an isomorphism between  $\mathcal{M}_{\mathcal{X}}$  and the stack of coherent pure 1-dimensional sheaves on  $X$  with determinant  $\mathcal{O}(E)$  and second Chern class  $-1$ . As shown in Section 4 of [3], this stack is isomorphic to  $\mathcal{R}_{\mathcal{X}}^{\text{big}}$ , which is isomorphic to the relative moduli stack of stable sheaves on the fibers of  $f : X \rightarrow \mathbf{P}^1$  of rank 1 and degree 1, and moreover  $\mathcal{R}_{\mathcal{X}}$  is isomorphic to  $X$  over  $\mathbf{P}^1$ . (Note that *loc. cit.* works over  $\mathbf{C}$  and only considers certain components of the moduli space.

However, the arguments there do not depend on the base field. If one is willing to believe that the moduli space fibers over  $\mathbf{P}^1$  by an elliptic fibration with smooth total space – following Lemma 5.12 below – an alternative argument to see minimality of the fibration is provided by appealing to Mukai’s results on the symplectic structure on the moduli space of sheaves on a K3 surface, Theorem 0.1 of [16]. This then implies that the moduli space is isomorphic to  $X$ , as desired.)

**Lemma 5.12.** *The stack  $\mathcal{M}_{\mathcal{X}}$  is a  $\mathbf{G}_m$ -gerbe over a smooth and separated scheme  $M_{\mathcal{X}}$  of dimension 2.*

*Proof.* Since  $\mathcal{M}_{\mathcal{X}}$  parametrizes stable sheaves, it is a  $\mathbf{G}_m$ -gerbe over its sheafification. Thus, the results will follow if we show that for any  $F \in \mathcal{M}_{\mathcal{X}}(k)$  the miniversal deformation space is of dimension 2. The scheme is separated because there is a unique stable limit by Langton’s theorem (for twisted sheaves, as explained in Lemma 2.3.3.2 of [7]).

Recall that there is an obstruction theory for  $F$  with values in

$$\ker(\mathrm{Tr} : \mathrm{Ext}^2(F, F) \rightarrow H^2(X, \mathcal{O}))$$

and a deformation theory with values in

$$\ker(\mathrm{Tr} : \mathrm{Ext}^1(F, F) \rightarrow H^1(X, \mathcal{O})).$$

By Serre duality, the obstruction theory is dual to the cokernel of the natural inclusion map

$$\Gamma(X, \mathcal{O}) \rightarrow \mathrm{Hom}(F, F),$$

which is trivial. The Riemann-Roch theorem shows that  $\chi(F, F) = 0$ , and it follows from stability (and Serre duality) that

$$\dim \mathrm{Ext}^1(F, F) = 2.$$

This shows that  $\mathcal{M}_{\mathcal{X}}$  is smooth, as desired.  $\square$

**Lemma 5.13.** *Pushforward defines an isomorphism*

$$\varphi : \mathcal{R}_{\mathcal{X}}^{\mathrm{big}} \rightarrow \mathcal{M}_{\mathcal{X}}$$

*of  $k$ -stacks.*

*Proof.* First we define the morphism. Fix a  $k$ -scheme  $T$ . A point of  $\mathcal{R}_{\mathcal{X}}$  is given by a lift  $T \rightarrow \mathbf{P}^1$  and a  $T$ -point as in Definition 5.9. But the stability condition is preserved under the pushforward

$$\mathcal{X} \times_{\mathbf{P}^1} T \rightarrow \mathcal{X} \times T$$

by definition of  $\mathcal{R}_{\mathcal{X}}$ . Hence, pushing forward along this morphism gives an object of  $\mathcal{M}_{\mathcal{X}}$ , giving the desired morphism.

To show that  $\varphi$  is an isomorphism of stacks, we will show that it is a proper monomorphism (hence a closed immersion) that is surjective on  $k$ -points (hence an isomorphism, as  $\mathcal{M}$  is smooth). We first make the following claim.

**Claim 5.14.** *Given a morphism  $a : T \rightarrow \mathbf{P}^1$  and a family  $F$  in  $\mathcal{R}_{\mathcal{X}}(T)$  with pushforward  $\iota_* F$  on  $\mathcal{X} \times T$ , we can recover the graph of  $a$  as the Stein factorization of the morphism*

$$\mathrm{Supp}(\iota_* F) \rightarrow \mathbf{P}^1 \times T,$$

*where  $\mathrm{Supp}(\iota_* F)$  denotes the scheme-theoretic support of  $\iota_* F$ .*

*Proof.* Since  $X \rightarrow \mathbf{P}^1$  is cohomologically flat in dimension 0, the claim follows if the natural map

$$\mathcal{O}_{\mathcal{X} \times_{\mathbf{P}^1} T} \rightarrow \text{End}(F)$$

is injective (as it is automatically compatible with base change on  $T$ ). By the assumption about the determinant of the fibers of  $F$ , for each geometric point  $t \rightarrow T$  we know that  $\mathcal{O}_{\mathcal{X}_t} \rightarrow \text{End}(F_t)$  is injective (as  $F_t$ , supported on one fiber of  $X \rightarrow \mathbf{P}^1$ , must have full support for the determinant on  $\mathcal{X}$  to be correct). The result now follows from Lemma 3.2.3 of [9].  $\square$

Suppose

$$t_1, t_2 : T \rightarrow \mathbf{P}^1$$

are two morphisms and  $F_i$  is an object of  $\mathcal{R}_{\mathcal{X}}(t_i)$  for  $i = 1, 2$ . Write

$$\iota_i : \mathcal{X} \times_{\mathbf{P}^1, t_i} T \rightarrow \mathcal{X} \times T$$

for the two closed immersions. If  $\varphi(F_1) \cong \varphi(F_2)$  then their scheme-theoretic supports agree, whence their Stein factorizations agree. By the Claim, the two maps  $t_1$  and  $t_2$  must be equal. But then  $F_1$  and  $F_2$  must be isomorphic because their pushforwards are isomorphic.

Now let us show that  $\varphi$  is proper. Fix a complete dvr  $R$  over  $k$  with fraction field  $K$ , a morphism  $\text{Spec } K \rightarrow \mathbf{P}^1$ , and an object  $F_K \in \mathcal{R}_{\mathcal{X}}(K)$ . Let  $\text{Spec } R \rightarrow \mathbf{P}^1$  be the unique extension ensured by the properness of  $\mathbf{P}^1$  and let

$$\iota : \mathcal{X} \otimes_{\mathbf{P}^1} R \rightarrow \mathcal{X} \otimes_k R$$

be the natural closed immersion. We wish to show that the unique stable limit  $F$  of  $\iota_* F_K$  has the form  $\iota_* F$  for an  $R$ -flat family of coherent  $\mathcal{X}_R$ -twisted sheaves (as the stability condition then follows by definition).

Let  $\mathcal{I}$  be the ideal of the image of  $\iota$ . By assumption, the map of sheaves

$$\nu : \mathcal{O}_{\mathcal{X} \otimes_k R} \rightarrow \mathcal{E}nd(F)$$

kills  $\mathcal{I}$  in the generic fiber over  $R$ . Since  $F$  is  $R$ -flat, so is  $\mathcal{E}nd(F)$  (as  $R$  is a dvr). Thus, the image of  $\mathcal{I}$  in  $\mathcal{E}nd(F)$  is  $R$ -flat. But this image has trivial generic fiber, hence must be trivial. It follows that  $\nu$  kills  $\mathcal{I}$ , whence  $F$  has a natural structure of pushforward along  $\iota$ , as desired.  $\square$

**Corollary 5.15.** *The morphism  $\mathcal{R}_{\mathcal{X}} \rightarrow \mathbf{P}^1$  is a  $\mathbf{G}_m$ -gerbe over a smooth surface that is flat over  $\mathbf{P}^1$ .*

*Proof.* Indeed, the stack  $\mathcal{R}_{\mathcal{X}}^{\text{big}}$  is smooth, whence  $\mathcal{R}_{\mathcal{X}}$  is a union of connected components in a smooth stack of dimension 2. The morphism  $\mathcal{R}_{\mathcal{X}} \rightarrow \mathbf{P}^1$  is dominant by definition and flat because  $\mathcal{R}_{\mathcal{X}}$  is integral.  $\square$

**Corollary 5.16.** *For any geometric point  $p \rightarrow \mathbf{P}^1$  and any object  $F$  of  $\mathcal{R}_{\mathcal{X}}(p)$ , there is a dvr  $A$ , a diagram*

$$\begin{array}{ccc} & p & \\ & \swarrow & \downarrow \\ \text{Spec } A & \longrightarrow & \mathbf{P}^1 \end{array}$$

whose horizontal arrow is dominant, and a family  $\mathcal{F} \in \mathcal{R}_{\mathcal{X}}(A)$  such that  $\mathcal{F}_p \cong F$ . In particular, any sheaf on a singular fiber sits in a flat family with a sheaf on the generic fiber.

*Proof.* This follows from the flatness of  $R_{\mathcal{X}} \rightarrow \mathbf{P}^1$ : one can take a general slice through the image of  $[F]$  and then take a finite normal covering to split the restriction of the  $\mathbf{G}_m$ -gerbe  $\mathcal{R}_{\mathcal{X}} \rightarrow R_{\mathcal{X}}$  to the slice.  $\square$

**Proposition 5.17.** *Suppose  $\mathcal{X}$  is a  $\mu_n$ -gerbe that deforms the trivial gerbe. The following hold for the moduli space  $R_{\mathcal{X}}$  and the  $\mathbf{G}_m$ -gerbe  $\mathcal{R}_{\mathcal{X}} \rightarrow R_{\mathcal{X}}$ .*

- (1) *The morphism*

$$R_{\mathcal{X}} \rightarrow \mathbf{P}^1$$

*is an étale form of the morphism*

$$X \rightarrow \mathbf{P}^1.$$

*In particular,  $R_{\mathcal{X}}$  is naturally an elliptic K3 surface.*

- (2) *The association  $\mathcal{X} \mapsto [R_{\mathcal{X}}]$  gives the image of the Brauer class of  $\mathcal{X}$  under the Artin-Tate isomorphism.*  
 (3) *The universal sheaf defines a Fourier-Mukai equivalence*

$$D^{-\text{tw}}(\mathcal{R}_{\mathcal{X}}) \xrightarrow{\sim} D^{\text{tw}}(\mathcal{X}).$$

*Proof of Proposition 5.17.* Let us first check the second statement. It suffices to verify this over the generic point  $\eta$  of  $\mathbf{P}^1$ , so that we may assume  $X$  and  $R$  are genus 1 curves over  $k(t)$ . The Leray spectral sequence and Tsen's theorem show that the edge map gives an isomorphism

$$\text{Br}(X_{\eta}) \xrightarrow{\sim} H^1(\eta, \text{Jac}(X_{\eta})),$$

which we can describe concretely as follows. Over  $\overline{k(t)}$  the gerbe  $\mathcal{X}_{\eta} \rightarrow X_{\eta}$  has trivial Brauer class, hence carries an invertible twisted sheaf  $\Lambda$  such that  $\Lambda^{\otimes n}$  has degree  $na(\mathcal{X})$ , which equals 0 by our assumption that  $\mathcal{X}$  deforms the trivial gerbe. Given an element  $\sigma$  of the Galois group of  $\overline{k(t)}$  over  $k(t)$ , there is an invertible sheaf  $L_{\sigma} \in \text{Pic}(X_{\overline{k(t)}/k(t)})$  such that  $\sigma^* \Lambda \otimes \Lambda^{\vee} \cong L_{\sigma}|_{\mathcal{X}_{\eta}}$ . This defines a 1-cocycle in the sheaf  $\text{Pic}_{X_{\eta}/\eta}$ , and its cohomology class is the image of a unique class in  $H^1(\eta, \text{Jac}(X_{\eta}))$ , as desired.

On the other hand, tensoring with  $\Lambda^{\vee}$  gives an isomorphism between the stack of invertible  $\mathcal{X}_{\eta}$ -twisted sheaves of degree 1 and the stack of invertible sheaves on  $X_{\eta}$  of degree 1. The latter stack is a gerbe over  $X_{\eta}$ , and the Galois group induces the cocycle given by the translation action of  $\text{Jac}(X_{\eta})$  on  $X_{\eta}$ . But this gives the edge map in the Leray spectral sequence. This proves the second statement.

Now let us show that  $R_{\mathcal{X}}$  is a form of  $X$  over  $\mathbf{P}^1$ . This turns out to be surprisingly subtle, and uses our assumption that  $\mathcal{X}$  deforms the trivial gerbe in an essential way. We begin with a lemma.

**Lemma 5.18.** *Given a fiber  $D \subset X$  of  $f$ , there is an invertible  $\mathcal{X} \times_X D$ -twisted sheaf  $\Lambda$  of rank 1 such that for each smooth curve  $C \rightarrow D$  the restriction of  $\Lambda$  to  $C$  has degree 0.*

*Proof.* We may replace  $D$  with its induced reduced structure, so we will assume that  $D$  is a reduced curve supported on a fiber of  $f$ . Write  $\pi : D \rightarrow \operatorname{Spec} k$  for the structure morphism. Let  $T$  be a smooth curve with two points 0 and 1 and  $\mathcal{Y} \rightarrow X_T$  a  $\mu_n$ -gerbe such that  $\mathcal{Y}_0 \cong B\mu_n$  and  $\mathcal{Y}_1 \cong \mathcal{X}$  (i.e., a curve connecting  $\mathcal{X}$  to the trivial gerbe). Let  $\mathcal{Z} \rightarrow D_T$  be the restriction to  $D$ . The gerbe  $\mathcal{Z}$  gives rise to a morphism of fppf sheaves

$$T \rightarrow \mathbf{R}^2\pi_*\mu_n.$$

The Kummer sequence shows that there is an isomorphism of sheaves

$$\operatorname{Pic}_{D/k} / n \operatorname{Pic}_{D/k} \xrightarrow{\sim} \mathbf{R}^2\pi_*\mu_n.$$

Thus, the gerbe  $\mathcal{Z}$  gives rise to a morphism

$$h : T \rightarrow \operatorname{Pic}_{D/k} / n \operatorname{Pic}_{D/k}$$

under which 0 maps to 0 (by assumption).

On the other hand, there is a multidegree morphism of  $k$ -spaces

$$\operatorname{Pic}_{D/k} \twoheadrightarrow \prod_{i=1}^m \mathbf{Z},$$

where  $m$  is the number of irreducible components of  $D$ . (This map comes from taking the degree of invertible sheaves pulled back to normalizations of components, and surjectivity is a basic consequence of the “complete gluing” techniques of [15].) This gives rise to a morphism

$$\deg_n : \operatorname{Pic}_{D/k} / n \operatorname{Pic}_{D/k} \rightarrow \prod_{i=1}^m \mathbf{Z} / n\mathbf{Z}$$

of sheaves. Composing with  $h$ , it follows from the connectedness of  $T$  that  $1 \in T(k)$  must map into the kernel of  $\deg_n$ .

By Tsen’s theorem, there is an invertible  $\mathcal{D}$ -twisted sheaf  $\Lambda$ , and the above calculation shows that  $\Lambda^{\otimes n}$  is the pullback of an invertible sheaf  $L$  on  $D$  such that for each irreducible component  $D_i \subset D$ , the pullback of  $L$  to the normalization of  $D_i$  has degree divisible by  $n$ . Let

$$\lambda_i \in \operatorname{Pic}(D_i)$$

be an invertible sheaf whose pullback to the normalization has degree  $-1$ . A simple gluing argument shows that there is an invertible sheaf

$$\lambda \in \operatorname{Pic}(D)$$

such that

$$\lambda|_{D_i} \cong \lambda_i$$

for each  $i$ . Replacing  $\Lambda$  by  $\Lambda \otimes \lambda$  yields an invertible  $\mathcal{D}$ -twisted sheaf whose restriction to each  $D_i$  has degree 0, yielding the desired result (as any non-constant  $C \rightarrow D$  factors through a  $D_i$ ).  $\square$



To show that  $R_{\mathcal{X}}$  is an étale form of  $X$ , we may base-change to the Henselization  $U$  of  $\mathbf{P}^1$  at a closed point. By Lemma 5.18, there is an invertible twisted sheaf  $\Lambda_u$  on the closed fiber

$$X_u \subset X_U$$

whose restriction to each irreducible component has degree 0. Since the obstruction to deforming such a sheaf lies in

$$H^2(X_u, \mathcal{O}) = 0,$$

we know that  $\Lambda_u$  deforms to an invertible  $\mathcal{X}_U$ -twisted sheaf  $\Lambda$  whose restriction to any smooth curve in any fiber of  $X_U$  over  $U$  has degree 0.

Tensoring with  $\Lambda$  gives an isomorphism of stacks

$$\mathrm{Sh}_{\mathcal{X}_U/U}(0, \mathcal{O}(E), -1) \xrightarrow{\sim} \mathrm{Sh}_{X_U/U}(0, \mathcal{O}(E), -1).$$

We claim that this isomorphism preserves  $H$ -stability. Since stability is determined by Hilbert polynomials, it suffices to prove the following.

**Claim 5.19.** *For any geometric point  $u \rightarrow U$  and any coherent  $\mathcal{X}$ -twisted sheaf  $G$ , the geometric Hilbert polynomial of  $\iota_* G$  equals the Hilbert polynomial of  $\Lambda^\vee \otimes G$ . In particular,  $G$  is stable if and only if  $\Lambda^\vee \otimes G$  is stable.*

*Proof.* Since  $G$  is filtered by subquotients supported on the reduced structure of a single irreducible component of  $D$ , it suffices to prove the result for such a sheaf. Let  $\nu : C \rightarrow D$  be the normalization of an irreducible component. The sheaves  $\nu_* \nu^* G$  and  $G$  differ by a sheaf of finite length. Thus, it suffices to prove the result for twisted sheaves on  $C$  and twisted sheaves of finite length. In either case, we are reduced to showing the following: given a finite morphism  $q : S \rightarrow X_u$  from a smooth  $\kappa(u)$ -variety, let  $\mathcal{S} \rightarrow S$  be the pullback of  $\mathcal{X}_u \rightarrow X_u$ . Then for any coherent  $\mathcal{S}$ -twisted sheaf  $G$ , the geometric Hilbert polynomial of  $q_* G$  equals the Hilbert polynomial of  $\Lambda^\vee \otimes q_* G$ .

Using the Riemann-Roch theorem for geometric Hilbert polynomials, the classical Riemann-Roch theorem, and the projection formula, we see that it is enough to prove that the geometric Hilbert polynomial of  $G$  (with respect to the pullback of  $H$  to  $S$ ) equals the usual Hilbert polynomial of  $\Lambda_{\mathcal{S}}^\vee \otimes G$ , under the assumption that  $\Lambda_{\mathcal{S}}$  has degree 0.

Let  $L \in \mathrm{Pic}^0(S)$  be the sheaf whose pullback to  $\mathcal{S}$  is isomorphic to  $\Lambda^{\otimes n}$ . Using the isomorphism between  $K(\mathcal{S}) \otimes \mathbf{Q}$  and  $K(X) \otimes \mathbf{Q}$ , the geometric Hilbert polynomial of  $G$  is identified with the usual Hilbert polynomial of the class

$$(\Lambda^\vee \otimes G) \otimes \frac{1}{n} L^\vee.$$

But  $L \in \mathrm{Pic}^0(S)$ , so this Hilbert polynomial is the same as the Hilbert polynomial of  $\Lambda^\vee \otimes G$ , as claimed.  $\square$

We conclude that  $R_{\mathcal{X}}$  is an étale form of the moduli space of stable sheaves on  $X \rightarrow \mathbf{P}^1$  of rank 1 and degree 1 on fibers (again using the assumption that  $\mathcal{X}$  deforms the trivial gerbe). This is isomorphic to  $X$  itself (see Example 5.11).

Finally, we need to prove that the universal sheaf defines an equivalence of derived categories. It is enough to show that the usual adjunction maps are quasi-isomorphisms (see, e.g., Proposition 3.3 of [13]). This shows that it is enough to establish the result étale-locally on  $\mathbf{P}^1$ . But then it is enough to show the result for an étale form of the problem, which means that it is enough to show that the ideal sheaf of the diagonal of  $X \times_{\mathbf{P}^1} X$  gives an equivalence of derived categories. The formula for the Fourier-Mukai transform shows that the ideal sheaf of the diagonal gives the identity map, which is an equivalence, as desired.  $\square$

*Remark 5.20.* As we saw in the proof of Lemma 5.18, the assumption in Proposition 5.17 that  $\mathcal{X}$  deforms the trivial gerbe yields a kind of homogeneity of degrees of restrictions of  $\mathcal{X}$  to components of fibers. This ensures that the resulting moduli problem can be compared with the classical stable sheaf theory on the underlying family of curves (étale-locally on the base). It is not fantastically clear to me at the present moment what happens without this hypothesis.

**Corollary 5.21.** *Suppose given a  $\mu_n$ -gerbe  $\mathcal{X} \rightarrow X_T$  over a connected base  $T$ , and a point  $t \in T(k)$  such that  $\mathcal{X}_t$  is trivial. The relative moduli stack*

$$\mathcal{R}_{\mathcal{X}}^{\text{big}} \rightarrow T$$

*contains an open substack*

$$\mathcal{R}_{\mathcal{X}} \rightarrow T$$

*whose geometric fiber over any  $t \rightarrow T$  satisfies the conclusion of Proposition 5.17. In particular, a gerbe  $\mathcal{X}$  on  $X \times T$  gives rise to a smooth family of surfaces  $R_{\mathcal{X}}$  over  $T$  with a morphism  $R_{\mathcal{X}} \rightarrow \mathbf{P}_T^1$  realizing each fiber as an étale form of  $X \rightarrow \mathbf{P}^1$ .*

*Proof.* By Lemma 5.12 and Lemma 5.13, the morphism  $\mathcal{R}_{\mathcal{X}}^{\text{big}} \rightarrow T$  is smooth of relative dimension 2. By smoothness, the functor of connected components of fibers is represented by an étale scheme  $C$  over  $T$ . The condition defining  $\mathcal{R}_{\mathcal{X}}$  is an open subset  $C' \subset C$ . By Proposition 5.17, every geometric fiber of  $C'$  is a singleton. It follows that  $C' \rightarrow T$  is an isomorphism, as desired.  $\square$

**Corollary 5.22.** *Given a field  $L/k$  and a  $\mu_n$ -gerbe  $\mathcal{X} \rightarrow X_L$  deforming the trivial gerbe, there is a natural isomorphism of numerical Chow groups*

$$\text{CH}(X) \otimes \mathbf{Q} \xrightarrow{\sim} \text{CH}(R_{\mathcal{X}}) \otimes \mathbf{Q}.$$

*In particular, if  $\text{Br}(L) = 0$  then any class in  $\text{Pic}(X_{\overline{L}})$  is defined over  $L$ .*

*Proof.* This follows from the cohomological form of Fourier-Mukai equivalence combined with the isomorphism in rational Chow theory

$$\text{CH}(\mathcal{X}) \otimes \mathbf{Q} \xrightarrow{\sim} \text{CH}(X) \otimes \mathbf{Q}$$

induced by pushforward.

To see that any class is algebraic, note that  $\text{Pic}_{R_{\mathcal{X}}/L}$  must have all of its points defined over separable extensions (as the Picard functor of a K3 surface is unramified). Thus, the points of

$\text{Pic}(R_{\mathcal{X}})$  are the Galois invariants in  $\text{Pic}(R_{\mathcal{X}} \otimes L^{\text{sep}})$ , and these points compute the Picard group by the assumption that  $\text{Br}(L) = 0$ .

By assumption, the rank of  $\text{CH}(R_{\mathcal{X}})$  is 24. It follows that the Galois action on

$$\text{Pic}(R_{\mathcal{X}} \otimes L^{\text{sep}}) \otimes \mathbf{Q}$$

is trivial, whence the action on the lattice  $\text{Pic}(R_{\mathcal{X}} \otimes L^{\text{sep}})$  is trivial, as desired.  $\square$

**Corollary 5.23.** *If  $X$  is a supersingular K3 surface of Artin invariant 10 then no elliptic pencil on  $X$  has a multisection of degree prime to  $p$ .*

*Proof.* First, suppose there is some genus 1 pencil  $\pi : X \rightarrow \mathbf{P}^1$  with a section. Since  $X$  has Artin invariant 10, any family

$$\mathcal{X} \rightarrow \text{Spec } k[[t]]$$

with supersingular generic fiber and special fiber will yield a restriction isomorphism

$$\text{Pic}(\mathcal{X}) \xrightarrow{\sim} \text{Pic}(X).$$

In particular, as in Lemma 2.3 of [12], any multisection of  $\pi$  will deform in any deformation of  $\pi$ .

In the situation of Proposition 5.17 applied to the universal element of  $\widehat{\text{Br}}(X)$ , we know that the geometric generic fiber pencil must be a non-trivial form of  $\pi$  (over  $k((t))$ ). It follows that the generic fiber pencil cannot have a multisection of degree prime to  $p$ . By the previous paragraph, we see that the special fiber thus cannot have a section, as claimed.  $\square$

## 6. Rational curves in moduli spaces

Fix a supersingular elliptic K3 surface  $X$  with Artin invariant 10 and let  $\tau : N \xrightarrow{\sim} \text{NS}(X)$  be a marking by the standard K3 lattice of Artin invariant 10. In addition, fix an elliptic pencil  $f : X \rightarrow \mathbf{P}^1$  and an ample divisor  $H \subset X$ . We will write  $\mathcal{P}$  for the period space of  $N$ -marked supersingular K3s defined by Ogus in [17].

The Leray spectral sequence for  $\mu_p$  with respect to  $f$  and the vanishing of cohomology of  $\mathbf{A}^1$  give a class

$$\tilde{\alpha} \in H^2(X \times \mathbf{A}^1, \mu_p)$$

that induces a closed and open immersion

$$\mathbf{A}^1 \hookrightarrow \mathbf{R}^2 \pi_* \mu_p$$

onto the connected component of the identity. Let  $\mathcal{X} \rightarrow X \times \mathbf{A}^1$  be a  $\mu_p$ -gerbe representing  $\tilde{\alpha}$ .

Every fiber of  $\mathcal{X}$  over  $\mathbf{A}^1$  deforms the trivial gerbe, since  $\mathcal{X}_0$  parametrizes the trivial class in  $H^2(X, \mu_p)$ , so we can apply Corollary 5.21. In particular, we can form the relative moduli space of stable twisted sheaves and use the fiberwise calculations of Section 5. Write  $Y := R_{\mathcal{X}} \rightarrow \mathbf{A}^1$  for the family of moduli spaces; there is a morphism  $Y \rightarrow \mathbf{P}^1 \times \mathbf{A}^1$  such that each fiber over  $\mathbf{A}^1$  is an étale form of  $X \rightarrow \mathbf{P}^1$ . Moreover, there is a canonical isomorphism  $Y_0 = X$ , giving a marking of  $Y_0$ .

**Proposition 6.1.** *The marking*

$$\tau : N \xrightarrow{\sim} \mathrm{NS}(Y_0)$$

*extends to a marking*

$$N \xrightarrow{\sim} \mathrm{Pic}_{Y/T},$$

*giving a family in  $\mathcal{P}(T)$ .*

*Proof.* By Corollary 5.22, we know that the fibers of  $Y$  over (not necessarily geometric) points of  $\mathbf{A}^1$  have Picard group of rank 22 (over their fields of definition) 22. On the other hand, for each point  $\mathrm{Spec} L \rightarrow \mathbf{A}^1$  with  $L$  of transcendence degree at most 1 over  $k$ , we have (by Tsen's theorem) that  $\mathrm{Pic}(Y_L)$  is the Galois-invariants in  $\mathrm{Pic}(Y_{\overline{L}})$ . Since the invariants have rank 22, it follows that the Galois action on  $\mathrm{Pic}(Y_{\overline{L}}) \otimes \mathbf{Q}$  is trivial, whereupon  $\mathrm{Pic}(Y_L) = \mathrm{Pic}(Y_{\overline{L}})$  by the semisimplicity of finite groups in characteristic 0.

Since  $Y_0$  has Artin invariant 10, the (injective) specialization map

$$\mathrm{Pic}(Y_{\overline{k((t))}}) \rightarrow \mathrm{Pic}(Y_0)$$

must be an isomorphism. By Popescu's theorem, this descends to some generically étale extension of  $\mathbf{A}^1$  with non-empty fiber over 0. Combining this with the previous statement, we see that the specialization map

$$\mathrm{Pic}(Y_{k(t)}) \rightarrow \mathrm{Pic}(Y_0)$$

over  $\mathbf{A}^1$  is an isomorphism. There is a canonical injection

$$\mathrm{Pic}(Y_{k(t)}) \rightarrow \mathrm{Pic}(Y)$$

given by taking closure of divisors, noting that every fiber is smooth, so that closure is a group homomorphism. This gives rise to a global marking

$$N \rightarrow \mathrm{Pic}(Y)$$

extending  $\tau$ , as desired. □

**Theorem 6.2.** *Given  $(X, \tau)$  and  $f : X \rightarrow \mathbf{P}^1$ ,  $\sigma : \mathbf{P}^1 \rightarrow X$  as above, let*

$$c : \mathbf{G}_a \rightarrow \mathcal{P}$$

*be the morphism induced by  $\varphi$  and Proposition 6.1, so that  $c(0) = (X, \tau)$ . The morphism  $c$  is non-constant. In particular, every point of Artin invariant 10 in  $\mathcal{P}_N$  lies in a non-trivial rational curve.*

*Proof.* It remains to prove that  $c$  is non-constant. But this follows from Corollary 5.2 and the fact that the geometric Brauer class over the generic point of  $(\mathbf{R}^2 \pi_* \mu_p)^\circ$  is not the pullback of a class over  $k$ . □

As we will see in the next section, distinct elliptic structures on  $X$  gives rise to physically distinct curves in  $\mathcal{P}$  (i.e., distinct even after reparametrization), showing that this construction yields an infinite collection of rational curves in  $\mathcal{P}$  through every general point. (It is somewhat surprising that this is true. One naively expects that the orbits of the automorphism group of  $X$  acting on the elliptic pencils should parametrize these special rational curves, but that turns out to be too pessimistic.)

## 7. Static pencils and deformations

Let  $X$  be a supersingular K3 surface of Artin invariant 10. Since the Artin invariant is maximal, for any deformation of  $X$  over a Henselian local ring, say

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} R, \end{array}$$

we have that the restriction map  $\mathrm{Pic}(\mathcal{X}) \rightarrow \mathrm{Pic}(X)$  is an isomorphism.

**Lemma 7.1.** *Suppose  $f : X \rightarrow \mathbf{P}^1$  is a pencil of genus 1 curves on  $X$ . Given a deformation  $\mathcal{X}/R$  as above, there is a deformation of  $f$  to a relative pencil  $F : \mathcal{X} \rightarrow \mathbf{P}_R^1$ .*

*Proof.* Let  $E$  be a smooth fiber of  $f$ . First, since  $X$  has Artin invariant 10, the class of  $E$  lifts over  $R$  to some invertible sheaf  $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ . It follows from basic deformation theory and the vanishing of  $H^1(X, \mathcal{O}(E))$  that in fact the global sections of  $\mathcal{L}|_X$  lift to sections of  $\mathcal{L}$  over  $\mathcal{X}$ . This lifts the pencil.  $\square$

**Definition 7.2.** Suppose  $\mathcal{X} \rightarrow R$  is a deformation of  $X$  over a Henselian local  $k$ -algebra  $k \rightarrow R$ . A pencil  $f : X \rightarrow \mathbf{P}^1$  is *static with respect to the deformation  $\mathcal{X}/R$*  if there is a lift of the pencil  $F : \mathcal{X} \rightarrow \mathbf{P}_R^1$  such that the pencils  $f \otimes R$  and  $F$  are isomorphic étale-locally on  $\mathbf{P}_R^1$ .

**Definition 7.3.** Two pencils  $f : X \rightarrow \mathbf{P}^1$  and  $g : X \rightarrow \mathbf{P}^1$  are *distinct* if there is no commutative diagram

$$\begin{array}{ccc} & X & \\ g \swarrow & & \searrow f \\ \mathbf{P}^1 & \xrightarrow{\quad} & \mathbf{P}^1 \end{array}$$

of isomorphisms. Equivalently, the fibers of  $f$  and  $g$  are not linearly equivalent.

**Definition 7.4.** Two pencils  $f : X \rightarrow \mathbf{P}^1$  and  $g : X \rightarrow \mathbf{P}^1$  are *transverse* if there is a smooth fiber  $E := f^{-1}(x)$  and a smooth fiber  $D := g^{-1}(x')$  such that  $E \cap D$  is reduced and  $\mathcal{O}(E) \not\cong \mathcal{O}(D)$ . In other words, general fibers of  $f$  and  $g$  are not linearly equivalent and they intersect transversely. Equivalently, the restriction of the morphism  $f$  to a general fiber of  $g$  is generically étale (or vice versa).

**Proposition 7.5.** *Suppose  $R$  is a Henselian local augmented  $k$ -algebra. If  $\mathcal{X} \rightarrow \mathrm{Spec} R$  is a deformation of  $X$  over which two transverse pencils remain static then  $\mathcal{X}$  is isomorphic to the constant deformation  $X_R$ .*

*Proof.* Write

$$f_1, f_2 : X \rightarrow \mathbf{P}^1$$

for the pencils with static lifts

$$F_1, F_2 : \mathcal{X} \rightarrow \mathbf{P}_R^1.$$

Write  $P = \mathbf{P}_R^1 \times \mathbf{P}_R^1$  and  $\widehat{P}$  for the formal scheme given by completing  $P$  along the augmentation ideal of  $R$ .

Since  $f_1$  and  $f_2$  are distinct, we have two finite maps

$$\Phi = (F_1, F_2) : \mathcal{X} \rightarrow P$$

and

$$\varphi = ((f_1)_R, (f_2)_R) : X_R \rightarrow P.$$

Moreover, since the pencils are transverse, we know that  $\Phi$  and  $\varphi$  are generically étale. Write  $\widehat{\mathcal{X}}$  and  $\widehat{X}_R$  for the formal schemes given by completing each along the augmentation ideal of  $R$ , so that there are finite generically étale morphisms of formal schemes

$$\widehat{\Phi} : \widehat{\mathcal{X}} \rightarrow \widehat{P}$$

and

$$\widehat{\varphi} : \widehat{X}_R \rightarrow \widehat{P}.$$

Under the assumption that both  $f_1$  and  $f_2$  are static pencils, we have that both  $\Phi$  and  $\varphi$  are isomorphic étale-locally on  $P$ . The étale sheaf  $\text{Isom}_P(\Phi, \varphi)$  is thus a torsor under the sheaf  $\text{Aut}(\varphi)$ , and similarly for  $\text{Isom}_{\widehat{P}}(\widehat{\Phi}, \widehat{\varphi})$ . By the infinitesimal rigidity of the étale site, we can identify the small étale site of  $\widehat{P}$  with the small étale site of  $P \otimes_R k$ . Via this identification, there is a reduction map

$$\text{Aut}_{\widehat{P}}(\varphi) \rightarrow \text{Aut}_{P_k}(\varphi_k).$$

Since  $X$  is integral and  $F$  is generically unramified, we know that this reduction map is an isomorphism (i.e., there are no infinitesimal automorphisms for a generically étale finite morphism with integral domain). There is thus an induced isomorphism

$$H^1(\widehat{P}_{\text{ét}}, \text{Aut}_{\widehat{P}}(\widehat{\varphi})) \xrightarrow{\sim} H^1((P_k)_{\text{ét}}, \text{Aut}_{P_k}(\varphi_k)).$$

Since  $\Phi$  and  $\varphi$  are isomorphic over  $k$  (being deformations of the same pair of pencils), it follows that the class of the torsor  $\text{Isom}_{\widehat{P}}(\widehat{\Phi}, \widehat{\varphi})$  is trivial. On the other hand, the Grothendieck existence theorem shows that the natural completion map

$$H^1(P_{\text{ét}}, \text{Aut}_P(\Phi, \varphi)) \rightarrow H^1((\widehat{P})_{\text{ét}}, \text{Aut}_{\widehat{P}}(\widehat{\Phi}, \widehat{\varphi}))$$

is injective (in fact, an isomorphism). It follows that  $\text{Isom}_P(\Phi, \varphi)$  is a trivial torsor, which shows that  $F$  is isomorphic to the trivial deformation, and thus that  $\mathcal{X}$  itself is isomorphic to the trivial deformation of  $X$ .  $\square$

**Definition 7.6.** Call two pencils  $f, g : X \rightarrow \mathbf{P}^1$  *inequivalent* if the fibers of  $f$  and  $g$  over 0 are not linearly equivalent.

**Proposition 7.7.** *Suppose  $R$  is a normal Henselian local augmented  $k$ -algebra. If  $\mathcal{X} \rightarrow \text{Spec } R$  is a deformation of  $X$  over which two inequivalent pencils remain static then  $\mathcal{X}$  is isomorphic to the constant deformation  $X_R$ .*

*Proof.* We start with the same morphisms

$$\Phi : \mathcal{X} \rightarrow P$$

and

$$\varphi : X_R \rightarrow P$$

as in the proof of Proposition 7.5, and, as above, we know that they are isomorphic étale-locally on  $P$ . What we do not know is that  $\Phi$  and  $\varphi$  are generically étale. We can avoid this since we are working with a normal domain  $R$  as the base ring.

First, write  $X \rightarrow P_k$  as a composition

$$X \rightarrow \overline{X} \rightarrow P_k$$

where  $\overline{X} \rightarrow P_k$  is separable and  $X \rightarrow \overline{X}$  is purely inseparable. This is canonical (taking  $\overline{X}$  to be the normalization of  $P$  inside the separable closure of  $k(P)$  inside  $k(X)$ ), and the factorization

$$X_R \rightarrow \overline{X}_R \rightarrow P$$

is identified with the normalization of  $P$  inside the separable closure of  $K(P)$  in  $K(X_R)$ .

Since  $\mathcal{X}$  is étale-locally isomorphic to  $X_R$ , it follows that for the analogous factorization

$$\mathcal{X} \rightarrow \overline{\mathcal{X}} \rightarrow P,$$

we know that  $\overline{\mathcal{X}}$  is an étale form of  $\overline{X}_R$ . By the arguments in the proof of Proposition 7.5, we conclude that  $\overline{\mathcal{X}} \cong \overline{X}_R$ . Choose an identification between the two. It remains to show that the two deformations of  $X \rightarrow \overline{X}$  are themselves isomorphic, knowing that they are simultaneously purely inseparable and étale-locally isomorphic.

Passing to the generic point of  $\overline{X}_R$ , we have two purely inseparable field extensions  $M_1/L$  and  $M_2/L$  such that

$$M_1 \otimes_L L^{\text{sep}} \cong M_2 \otimes_L L^{\text{sep}}.$$

Since every scheme in sight is normal, it suffices to show that  $M_1 \cong M_2$  (as  $L$ -algebras). But the automorphism sheaf  $\text{Aut}_L(M_1)$  on the small étale site of  $\text{Spec } L$  is the singleton sheaf (since  $M_1$  and  $L^{\text{sep}}$  are linearly disjoint, and a purely inseparable field extension has trivial automorphism group). Thus, étale forms are all trivial, as desired.  $\square$

*Remark 7.8.* The reader will note the curious fact that the proof of Proposition 7.7 uses the normality of  $R$  in an essential way. In particular, we gain no insight into the infinitesimal properties of pairs of static pencils that are not transverse. As Maulik as pointed out to us, if one works over a finite base field, one can deduce from Propositions 7.5 and 7.7 that given an infinite list of pencils on  $X$ , applying the construction of Theorem 6.2 below yields an infinite list of curves such that for any given finite order  $n$ , infinitely many of these curves must agree up to order  $n$  (as there are only finitely many jets of a given order on the Ogus space over a finite field). In particular, we cannot have infinitely many pairwise transverse pencils on a supersingular K3 surface of Artin invariant 10.

**Proposition 7.9.** *Let  $X$  be a K3 surface and  $\pi : X \rightarrow \mathbf{P}^1$  an elliptic pencil. The locus in  $\text{Def}_X$  parametrizing deformations over which  $\pi$  remains static is 1-dimensional.*

*Proof.* Let  $\text{Def}_{X/\mathbf{P}^1}$  be the functor whose objects over an augmented Artinian  $k$ -algebra  $A$  are Cartesian diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbf{P}_k^1 & \longrightarrow & \mathbf{P}_A^1 \end{array}$$

in which the vertical arrows are flat and proper. (In other words,  $\text{Def}_{X/\mathbf{P}^1}$  parametrizes relative deformations of the pencil.) Let  $\text{Def}_{X/\mathbf{P}^1}^s$  denote the subfunctor parametrizing families that are isomorphic to the constant family

$$\begin{array}{ccc} X & \longrightarrow & X \times \mathbf{P}_A^1 \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \longrightarrow & \mathbf{P}_A^1 \end{array}$$

locally on  $\mathbf{P}_A^1$ .

**Lemma 7.10.** *The functor  $\text{Def}_{X/\mathbf{P}^1}^s$  is prorepresentable.*

*Proof.* We will temporarily write  $F$  for the functor  $\text{Def}_{X/\mathbf{P}^1}^s$ . To show that  $F$  is prorepresentable, we will use Schlessinger's criterion. Given morphisms  $A \rightarrow C$  and  $B \rightarrow C$  in  $\text{Art}_k$ , there is a natural diagram

$$(7.0.2) \quad F(A \times_C B) \rightarrow F(A) \times_{F(C)} F(B)$$

We need to check the following.

- (1) (7.0.2) is a surjection when  $B \rightarrow C$  is small
- (2) (7.0.2) is a bijection when  $C = k$  and  $B = k[\varepsilon]$
- (3)  $F(k[\varepsilon])$  is a finite-dimensional vector space (with its natural structure)
- (4) if  $A \rightarrow C$  is small then  $F(A \times_C A) \rightarrow F(A) \times_{F(C)} F(A)$  is a bijection

Since we already know that these conditions hold for the moduli of diagrams  $X \rightarrow \mathbf{P}^1$ , the key is showing that they respect the étale-local triviality condition. In other words, we need to show that given a family

$$\mathcal{X} \rightarrow \mathbf{P}_{A \times_C B}^1$$

such that the restricted families

$$\mathcal{X}_A \rightarrow \mathbf{P}_A^1$$

and

$$\mathcal{X}_B \rightarrow \mathbf{P}_B^1$$

are étale-locally isomorphic to the trivial family, then the same holds for the original family. But we know that the morphism of  $\mathbf{P}_{A \times_C B}^1$ -schemes

$$\text{Isom}_{\mathbf{P}_{A \times_C B}^1}(\mathcal{X}, X_{A \times_C B}) \rightarrow \text{Isom}_{\mathbf{P}_A^1}(\mathcal{X}_A, X_A) \times_{\text{Isom}_{\mathbf{P}_C^1}(\mathcal{X}_C, X_C)} \text{Isom}_{\mathbf{P}_B^1}(\mathcal{X}_B, X_B)$$

is an isomorphism under all of the listed conditions because the stack of elliptic surfaces is algebraic. The results now follow from the topological invariance of the étale site.  $\square$



**Lemma 7.11.** *The formal scheme  $\mathrm{Def}_{X/\mathbf{P}^1}^s$  is formally smooth and 1-dimensional.*

*Proof.* The infinitesimal automorphism sheaf  $\mathcal{A}$  of  $X \rightarrow \mathbf{P}^1$  is precisely the normal sheaf of the 0-section  $\mathbf{P}^1 \rightarrow J$  of the Jacobian fibration of  $X$ . Since  $J$  is also a K3 surface, the normal sheaf is  $\mathcal{O}(-2)$ . In particular,  $H^1(\mathbf{P}^1, \mathcal{A})$  is 1-dimensional and  $H^2(\mathbf{P}^1, \mathcal{A}) = 0$ . Given a square-zero extension  $A \rightarrow A_0$  with ideal sheaf  $I$  and a point  $\mathrm{Def}_{X/\mathbf{P}^1}^s(A_0)$ , the lifts to  $A$  are obstructed by elements of  $H^2(\mathbf{P}_{A_0}^1, \mathcal{A} \otimes I)$  and form a pseudo-torsor under  $H^1(\mathbf{P}_{A_0}^1, \mathcal{A} \otimes I)$ . It follows that deformations are unobstructed (so  $\mathrm{Def}_{X/\mathbf{P}^1}^s$  is formally smooth) and the tangent space is 1-dimensional, as desired.  $\square$

**Lemma 7.12.** *The forgetful morphism*

$$\mathrm{Def}_{X/\mathbf{P}^1}^s \rightarrow \mathrm{Def}_X$$

*is a closed immersion of formal  $k$ -schemes*

*Proof.* It is enough to show that the tangent map

$$T \mathrm{Def}_{X/\mathbf{P}^1}^s \rightarrow T \mathrm{Def}_X$$

is injective. Suppose

$$X_\varepsilon \rightarrow \mathbf{P}_{k[\varepsilon]}^1$$

is a tangent vector that maps to 0. This means that the underlying surface  $X_\varepsilon$  is isomorphic to  $X_{k[\varepsilon]}$  in a way compatible with the identifications over  $k$ . Choosing such an isomorphism yields two morphisms

$$f, g : X_{k[\varepsilon]} \rightarrow \mathbf{P}_{k[\varepsilon]}^1$$

with the property that for each  $k$ -point  $p \in \mathbf{P}^1$ , the restrictions of  $f$  and  $g$  to  $p \otimes_k k[\varepsilon]$  are constant. Consider the Stein factorization of the induced morphism

$$(f, g) : X_{k[\varepsilon]} \rightarrow \mathbf{P}_{k[\varepsilon]}^1 \times_{\mathrm{Spec} k[\varepsilon]} \mathbf{P}_{k[\varepsilon]}^1.$$

Since every scheme in sight is  $\varepsilon$ -flat, the Stein factorization is a finite  $\varepsilon$ -flat morphism

$$S \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

over  $k[\varepsilon]$ . Moreover,  $S \otimes_{k[\varepsilon]} k$  is isomorphic to the diagonal by the definition of the moduli problems. Thus,  $S$  is an infinitesimal deformation of the diagonal  $\Delta \subset \mathbf{P}_k^1 \times \mathbf{P}_k^1$ . We know that  $\Delta^2 = 2$ , so the space of infinitesimal deformations is a torsor under  $H^0(\mathbf{P}^1, \mathcal{O}(2))$ . In fact, this is just the tangent space to the automorphism group scheme  $\mathrm{PGL}_2$  of  $\mathbf{P}^1$ . Since each  $k$ -point must be fixed (by the static assumption), there are no non-trivial infinitesimal automorphisms, and we see that  $f = g$ , as desired.  $\square$

This completes the proof of Proposition 7.9.  $\square$

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